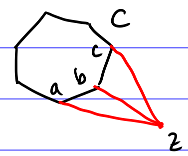
Q6. G nonbipartite, Δ-free simple graph n vertices $\delta(G) = k$

ℓ is min length of an odd cycle

a)



Suppose $\exists\ z$ with 3 neighbours in C.
Then it forms three cycles

$zab \qquad zbc \qquad zca$

Assume a, b, c appear in order along the cycle. Let $d(a,b)$ represent the # of edges between a and b in the cyclic order

$$d(a,b) + d(b,c) + d(c,a) = \ell$$

At least one them must be odd, since ℓ is odd. WLOG let $d(a,b)$ be odd. Then zab is an

So we must have $2 + d(a,b) \geq \ell = d(a,b) + d(b,c) + d(c,a)$

This is only possible if $d(a,b) = d(b,c) = 1 \Rightarrow 2$ triangles. This is a contradiction.

Note: The oddness of ℓ is not that critical here., so I dont think its required. So let ℓ be a minimal cycle of any length.

Then the cycles have length

$$2 + d(a,b) \qquad 2 + d(b,c) \qquad 2 + d(c,a)$$

They all have length at least ℓ (by definition of ℓ)

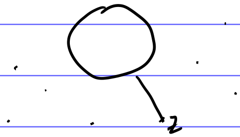
$$\Rightarrow 2 + d(a,b) + 2 + d(b,c) + 2 + d(c,a) \geq 3\ell$$

But this gives us

$$6 + \ell \geq 3\ell \quad \Rightarrow \quad \ell \leq 3.$$

But our graph is triangle free, so this is a contradiction.
(we cannot have $\ell = 2$ or $\ell = 1$ since the graph is simple)

b)



\# of edges between $V(C)$ and $V(G) - V(C)$.

$$= \sum_{v \in V(C)} d(v) - \overbrace{2}^{\#\text{ of edges within } V(C)} = |V(C)|\,(k-2) = \ell(k-2)$$

Since each $z \in V(G) - V(C)$ can be connected to at most 2 $V(C)$ vertices we also have the upper bound

$$\le 2\,|V(G) - V(C)| = 2(n-\ell)$$

$$\Rightarrow \quad \ell(k-2) \le 2(n-\ell) \quad \Rightarrow \quad \frac{k\ell}{2} \le n$$

Q7 a) All vertices have even degree.

Prob 1.2.27 : Every even graph decomposes into cycles.

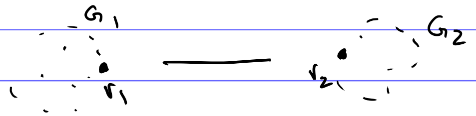
$G = C_1 \sqcup C_2 \ \cdots \ \sqcup C_k$

Take any edge, it must belong to one of the cycles $\Rightarrow$ it cannot be a cut edge

b) $v_1 \cdots v_J$ $\quad J-2$

v_{J+1} $\quad v_{J+2}$ $\quad$ IB obvious.

c) Take 2 copies of a graph



G_1 has the following props

1. $d(r_1) = 2k$
2. $d(v) = 2k+1 \quad v \in G - r_1$

Connect r_1 and r_2. Now $d(r_1) = d(r_2) = 2k+1$. $r_1 r_2$ is a cut edge.

How to construct G_1?

Form a $2k$ regular graph with $2k+2$ vertices. $v_1 \cdots v_{2k}, v_{2k+1}, v_{2k+2}$ where the last two vertices are not connected to each other — this by part (b) and an obvious induction.

Create r_1 and connect it to $2k$ vertices, and connect the 2 remaing vertices v_{2k+1}, v_{2k+2}.

$deg(r_1) = 2k \quad , \quad d(v_i) = 2k+1$.

$\blacksquare$